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Axially Symmetric Solutions in General Scalar–Tensor Theory

S. Chatterjee

Department of Physics, B.B. College, Asansol — 713303, India

A. Banerjee

Department of Physics, The University of Jadavpur, Calcutta — 700032, India

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We generalise Ernst's derivation of the axially symmetric solutions of Einstein's field equations to the general scalar–tensor theory proposed by Nordtvedt. The solution of the Nordtvedt theory differs by a conformal transformation from the Brans–Dicke solution. The Kerr-like solution of the Nordtvedt theory is obtained as an example.

1. INTRODUCTION

Einstein's field equations being not completely Machian in nature, Brans and Dicke (1961) introduce a scalar field to make things less reliant on the absolute properties of space. But current experimental evidence suggests that if B–D theory is to be justified the value of the parameter ω in this theory should be very large, in which case, however, there does not remain much difference between the two theories. Since there is no *a priori* reason to exclude the introduction of any long-range scalar field, Nordtvedt (1970) proposes a modification of the B–D theory where ω becomes a function of the scalar field φ.

With the line element

$$ds^2 = f(dt - \Omega\, d\varphi)^2 - f^{-1}\left[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2\, d\varphi^2\right]$$

where f, γ, and Ω depend on (ρ, z) only, Ernst (1968) gives a very elegant and compact method of finding the solution of stationary axisymmetric

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distribution in empty space from the knowledge of a complex function ξ satisfying

$$\mathrm{Re}(\xi)\nabla^2\xi = \nabla\xi\cdot\nabla\xi \tag{1}$$

where $\xi = f + i\psi$ and $\rho^{-1}f^2\nabla\Omega = \hat{n}\times\nabla\psi$ where $\hat{n}$ is a unit vector in the φ direction. The authors (1980) in an earlier paper extend the Ernst procedure to the B–D theory. In this paper the above procedure is further extended to Nordtvedt's theory, where, interestingly, it is found that with the conformal transformation proposed by Dicke (1962) the solutions as obtained from both B–D and Nordtvedt theories are virtually the same. This similarity in the conformal frame, however puzzling it may seem at first sight, is only apparent, because completely new solutions are found when we go back to the original metric.

2. FIELD EQUATIONS

The field equations in the Nordtvedt theory are given by

$$\varphi\left(R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R\right) = 8\pi T_{\mu\nu} + (\omega/\varphi)\left(\varphi_\mu\varphi_\nu - \tfrac{1}{2}g_{\mu\nu}\varphi_k\varphi^k\right) + \varphi_{\mu;\nu} - g_{\mu\nu}\Box\varphi \tag{2}$$

$$\Box\varphi = \frac{1}{3+2\omega}\left(8\pi T - \varphi_\lambda\varphi^\lambda\frac{d\omega}{d\varphi}\right) \tag{3}$$

With the same set of transformation relations pointed out by Dicke it can be shown that in the matter-free space the Nordtvedt's general scalar–tensor theory can also be reduced after some straightforward calculations to

$$\bar{R}_{\mu\nu} = -\left(\omega + \tfrac{3}{2}\right)\Phi_\mu\Phi_\nu \tag{4}$$

where

$$\bar{g}_{\mu\nu} = \varphi g_{\mu\nu},\ \Phi = \ln\varphi$$

The wave equation becomes

$$\bar{\Box}\Phi = \bar{g}^{\mu\nu}\Phi_{;\mu\nu} = -\frac{1}{2\omega+3}\left(\Phi_\lambda\Phi^\lambda\frac{d\omega}{d\Phi}\right) \tag{5}$$

and $\bar{R}_{\mu\nu}$ is the Ricci tensor and $\bar{\Box}$ is the D'Alembertian formed out of $\bar{g}_{\mu\nu}$.

In the case of axial symmetry this conformal transformation has the added advantage that with $R_3^3 + R_4^4 = 0$ one can use two metric components instead of the usual three, thus considerably simplifying the field equations.

In the rest of the paper the details of the field equations are omitted since they are discussed in the author's earlier paper referred to above. From the field equations it follows that of all the metric components, f and Ω would remain the same in Einstein's, Brans–Dicke's, and Nordtvedt's theories. For the calculation of the rest of the metric we here adopt a slightly different procedure. The field equations, in general, owing to their nonlinearity have so far proven too complicated to yield solutions except in a few special cases. However, as pointed out by several authors, they possess a large amount of hidden symmetry. Given one solution we may use certain transformations to generate other solutions. With this motivation we write

$$\gamma = \gamma_k + \tilde{\gamma} \tag{6}$$

where γ_k is the value of γ for some known metric, $\tilde{\gamma}$ is some extra variable, and f and Ω keep their forms for this known metric. From the field equations we get

$$\tilde{\gamma}_1 = \frac{(\omega + \frac{3}{2})}{2}\rho(\Phi_1{}^2 - \Phi_2{}^2) \tag{7a}$$

$$\tilde{\gamma}_2 = (\omega + \tfrac{3}{2})\rho\Phi_1\Phi_2 \tag{7b}$$

where suffix 1 refers to $\partial/\partial\rho$ and 2 to $\partial/\partial z$ and the integrability condition for $\tilde{\gamma}$ is satisfied via the field equations. From equation (7) it is evident that the geometry within the group orbits (generated by $\partial/\partial t$ and $\partial/\partial\varphi$) is left unchanged. Only the complementary (ρ, z) surfaces are deformed.

From the wave equation we get

$$g''\left(\Phi_{11} + \Phi_{22} + \frac{\Phi_1}{\rho}\right) = \frac{1}{3+2\omega}g^{11}(\Phi_1{}^2 + \Phi_2{}^2)\frac{d\omega}{d\Phi} \tag{8}$$

which is also a consequence of the field equations. Thus we are left with two independent equations and three unknowns $(\tilde{\gamma}, \Phi, \omega)$ forcing us to assume an equation between any two of them. In our case we would assume a functional relationship between Φ and ω. When equations (7) and (8) are solved the complete metric is determined through (6).

We now make a transformation to the prolate spheroidal coordinates (x, y) through

$$\rho = a(x^2 - 1)^{1/2}(1 - y^2)^{1/2}, \qquad z = axy \tag{9}$$

In the spheroidal coordinates the metric remains axially symmetric even when it is made to depend on any one of the two variables (x, y). A relevant solution may be when Φ and ω are functions of x only. For the stationary case, the wave equation becomes in the new coordinates

$$(x^2-1)\Phi_{xx}+2x\Phi_x+\frac{x^2-1}{2\omega+3}\Phi^2{}_x\frac{d\omega}{d\Phi} \tag{10}$$

which on first integration becomes

$$\Phi_x=\frac{A}{(x^2-1)(2\omega+3)^{1/2}} \tag{11}$$

where A is an integration constant. Using (11) and (7a, 7b) we get

$$\tilde{\gamma}=\frac{A^2}{8}\ln\frac{x^1-1}{x^2-y^2} \tag{12}$$

Thus the solution is asymptotically flat. Moreover, when the constant A connected with the scalar field vanishes, $\tilde{\gamma}=0$ and we get back the Einstein solution in vacuum.

To get Φ we make the simple choice

$$\omega=\frac{1-3(\Phi-C)^2}{2(\Phi-C)^2} \tag{13}$$

where C is a constant.

Now we get from equation (11) after adjusting constants

$$\Phi=C\left[\left(\frac{x+1}{x-1}\right)^{-A/2}-1\right] \tag{14}$$

In the asymptotic region when $x\rightarrow\infty$,

$$\Phi\rightarrow 0 \qquad \text{and} \qquad \varphi=e^{\Phi}\rightarrow 1$$

3. DISCUSSION

As pointed out earlier, the only change in the metric due to the introduction of the scalar field φ and variability of ω with the φ is the addition of the extra term $\tilde{\gamma}$. But from equation (12) we find that $\tilde{\gamma}$ is independent of ω. Thus when $\omega\rightarrow\infty$, the metric remains unaltered. This is rather puzzling because when ω becomes large all solutions in B–D theory reduce to Einstein's. Moreover, the proposed variation of ω with Φ does not appear to have any influence on the metric itself, resulting in exactly the same solution in both B–D and Nordtvedt theories. This is rather paradoxical. But the paradox resolves itself when one realizes that this apparent inconsistency develops only when one makes this conformal transformation. When we go back to the original unbarred system we get

$$\varphi^{-1}e^{\bar{\gamma}} = e^{-\Phi}e^{\bar{\gamma}} = \exp C\left[1 - \left(\frac{x+1}{x-1}\right)^{-A/2}\right]\left(\frac{x^2-1}{x^2-y^2}\right)^{A^2/4} \tag{15}$$

and this is a completely new solution.

Example. Nordtvedt–Kerr solution: We define a new variable $\xi = (\zeta - 1)/(\zeta + 1)$ and use $\zeta = x\cos\lambda + iy\sin\lambda$ as a solution of the differential equation (3), where λ is a constant. When the entire metric is constructed we get the Kerr solution as

$$\begin{aligned} ds^2 = &\left[(r^2 + a^2\cos^2\theta)\left(d\theta^2 - \frac{dr^2}{r^2 + a^2 - 2mr}\right)\right] \\ &+ \left[(r^2 + a^2)\sin^2\theta\, d\varphi^2 - dt^2 \right. \\ &\left. + \frac{2mr}{r^2 + a^2\cos^2\theta}\left(dt + a\sin^2\theta\, d\varphi\right)^2\right] \end{aligned} \tag{16}$$

where the coordinates are defined by

$$r = x(m^2 - a^2)^{1/2} + m, \qquad y = \cos\theta$$

New solutions which may be termed *vacuum Kerr–Nordtvedt solutions* can then be obtained by multiplying the $(r^2 + a^2\cos^2\theta)$ section of the Kerr metric by $e^{2\bar{\gamma}}$ where $\bar{\gamma}$ can be found out as discussed in the previous section and keeping the rest of the metric unchanged.

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